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Abstract. Let p be a prime number and let n be a non-zero natural number. We compute the descending Loewy series of the algebra $\mathcal{R}_n/p\mathcal{R}_n$, where $\mathcal{R}_n$ denotes the ring of virtual ordinary characters of the symmetric group $\mathfrak{S}_n$.

Let p be a prime number and let n be a non-zero natural number. Let $\mathbb{F}_p$ be the finite field with p elements and let $\mathfrak{S}_n$ be the symmetric group of degree n. Let $\mathcal{R}_n$ denote the ring of virtual ordinary characters of the symmetric group $\mathfrak{S}_n$ and let $\bar{\mathcal{R}}_n = \mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{R}_n = \mathcal{R}_n/p\mathcal{R}_n$. The aim of this paper is to determine the descending Loewy series of the $\mathbb{F}_p$-algebra $\bar{\mathcal{R}}_n$ (see Theorem A). In particular, we deduce that the Loewy length of $\bar{\mathcal{R}}_n$ is $[n/p]+1$ (see Corollary B). Here, if x is a real number, $[x]$ denotes the unique $r \in \mathbb{Z}$ such that $r \leqslant x < r+1$.

Let us introduce some notation. If $\varphi \in \mathcal{R}_n$, we denote by $\bar{\varphi}$ its image in $\bar{\mathcal{R}}_n$. The radical of $\bar{\mathcal{R}}_n$ is denoted by $\operatorname{Rad}\bar{\mathcal{R}}_n$. If X and Y are two subspaces of $\bar{\mathcal{R}}_n$, we denote by XY the subspace of $\bar{\mathcal{R}}_n$ generated by the elements of the form xy, with $x \in X$ and $y \in Y$.

Compositions, partitions. A *composition* is a finite sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of non-zero natural numbers. We set $|\lambda| = \lambda_1 + \cdots + \lambda_r$ and we say that λ is a *composition of* $|\lambda|$. The λ_i's are called the *parts* of λ. If moreover $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_r$, we say that λ is a *partition of* $|\lambda|$. The set of compositions (resp. partitions) of n is denoted by $\operatorname{Comp}(n)$ (resp. $\operatorname{Part}(n)$). We denote by $\hat{\lambda}$ the partition of n obtained from λ by reordering its parts. So $\operatorname{Part}(n) \subset \operatorname{Comp}(n)$ and $\operatorname{Comp}(n) \to \operatorname{Part}(n)$, $\lambda \mapsto \hat{\lambda}$ is surjective. If $1 \leqslant i \leqslant n$, we denote by $r_i(\lambda)$ the number of occurences of i as a part of λ. We set

$$\pi_p(\lambda) = \sum_{i=1}^{n} \Big[\frac{r_i(\lambda)}{p}\Big].$$

Recall that λ is called *p-regular* (resp. *p-singular*) if and only if $\pi_p(\lambda) = 0$ (resp. $\pi_p(\lambda) \geqslant 1$). Note also that $\pi_p(\lambda) \in \{0, 1, 2, \dots, [n/p]\}$ and that $\pi_p(\hat{\lambda}) = \pi_p(\lambda)$. Finally, if $i \geqslant 0$, we set

$$\operatorname{Part}_i^{(p)}(n) = \{\lambda \in \operatorname{Part}(n) \mid \pi_p(\lambda) \geqslant i\}.$$

Young subgroups. For $1 \leqslant i \leqslant n-1$, let $s_i = (i, i+1) \in \mathfrak{S}_n$. Let $S_n = \{s_1, s_2, \dots, s_{n-1}\}$. Then $(\mathfrak{S}_n, S_n)$ is a Coxeter group. We denote by $\ell : \mathfrak{S}_n \to \mathbb{N}$ the associated length function. If $\lambda = (\lambda_1, \dots, \lambda_r) \in \operatorname{Comp}(n)$, we set

$$S_\lambda = \{s_i \mid \forall\ 1 \leqslant j \leqslant r,\ i \neq \lambda_1 + \cdots + \lambda_j\}.$$

Let $\mathfrak{S}_\lambda =< S_\lambda >$. Then $(\mathfrak{S}_\lambda, S_\lambda)$ is a Coxeter group: it is a standard parabolic subgroup of $\mathfrak{S}_n$ which is canonically isomorphic to $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r}$. Note that

(1) *$\mathfrak{S}_\lambda$ and $\mathfrak{S}_\mu$ are conjugate in $\mathfrak{S}_n$ if and only if $\hat{\lambda} = \hat{\mu}$.*

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We write $\lambda \subset \mu$ if $\mathfrak{S}_\lambda \subset \mathfrak{S}_\mu$ and we write $\lambda \leqslant \mu$ if $\mathfrak{S}_\lambda$ is contained in a subgroup of $\mathfrak{S}_n$ conjugate to $\mathfrak{S}_\mu$. Then $\subset$ is an order on $\mathrm{Comp}(n)$ and $\leqslant$ is a preorder on $\mathrm{Comp}(n)$ which becomes an order when restricted to $\mathrm{Part}(n)$.

Let $X_\lambda = \{w \in \mathfrak{S}_n \mid \forall\, x \in \mathfrak{S}_\lambda,\ \ell(wx) \geqslant \ell(w)\}$. Then X_λ is a cross-section of $\mathfrak{S}_n/\mathfrak{S}_\lambda$. Now, let $\mathcal{N}_\lambda = N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda)$ and $W(\lambda) = \mathcal{N}_\lambda \cap X_\lambda$. Then $W(\lambda)$ is a subgroup of $\mathcal{N}_\lambda$ and $\mathcal{N}_\lambda = W(\lambda) \ltimes \mathfrak{S}_\lambda$. Note that

$$W(\lambda) \simeq \mathfrak{S}_{r_1(\lambda)} \times \cdots \times \mathfrak{S}_{r_n(\lambda)}. \tag{2}$$

Recall that, for a finite group G, the *p-rank* of G is the maximal rank of an elementary abelian p-subgroup of G. For instance, $[n/p]$ is the p-rank of $\mathfrak{S}_n$. So

$$\pi_p(\lambda) \textit{ is the } p\textit{-rank of } W(\lambda). \tag{3}$$

If λ, $\mu \in \mathrm{Comp}(n)$, we set

$$X_{\lambda\mu} = (X_\lambda)^{-1} \cap X_\mu.$$

Then $X_{\lambda\mu}$ is a cross-section of $\mathfrak{S}_\lambda\backslash\mathfrak{S}_n/\mathfrak{S}_\mu$. Moreover, if $d \in X_{\lambda\mu}$, there exists a unique composition ν of n such that $\mathfrak{S}_\lambda \cap {}^d\mathfrak{S}_\mu = \mathfrak{S}_\nu$. This composition will be denoted by $\lambda \cap {}^d\mu$ or by ${}^d\mu \cap \lambda$.

The ring $\mathcal{R}_n$. If $\lambda \in \mathrm{Comp}(n)$, we denote by 1_λ the trivial character of $\mathfrak{S}_\lambda$ and we set $\varphi_\lambda = \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} 1_\lambda$. Then, by (1), we have $\varphi_\lambda = \varphi_{\hat{\lambda}}$. We recall the following well-known old result of Frobenius:

$$(\varphi_\lambda)_{\lambda \in \mathrm{Part}(n)} \textit{ is a } \mathbb{Z}\textit{-basis of } \mathcal{R}_n. \tag{4}$$

Moreover, by the Mackey formula for tensor product of induced characters, we have

$$\varphi_\lambda\varphi_\mu = \sum_{d \in X_{\lambda\mu}} \varphi_{\lambda\cap{}^d\mu} = \sum_{d \in X_{\lambda\mu}} \varphi_{\widehat{\lambda\cap{}^d\mu}}. \tag{5}$$

Let us give another form of (5). If $d \in X_{\lambda\mu}$, we define $\Delta_d : \mathcal{N}_\lambda \cap {}^d\mathcal{N}_\mu \to \mathcal{N}_\lambda \times \mathcal{N}_\mu$, $w \mapsto (w, d^{-1}wd)$. Let $\bar{\Delta}_d : \mathcal{N}_\lambda \cap {}^d\mathcal{N}_\mu \to W(\lambda) \times W(\mu)$ be the composition of Δ_d with the canonical projection $\mathcal{N}_\lambda \times \mathcal{N}_\mu \to W(\lambda) \times W(\mu)$. Then the kernel of $\bar{\Delta}_d$ is $\mathfrak{S}_{\lambda\cap{}^d\mu}$, so $\bar{\Delta}_d$ induces an injective morphism $\tilde{\Delta}_d : W(\lambda,\mu,d) \hookrightarrow W(\lambda) \times W(\mu)$, where $W(\lambda,\mu,d) = (\mathcal{N}_\lambda \cap {}^d\mathcal{N}_\mu)/\mathfrak{S}_{\lambda\cap{}^d\mu}$. Now, $W(\lambda) \times W(\mu)$ acts on $\mathfrak{S}_\lambda\backslash\mathfrak{S}_n/\mathfrak{S}_\mu$ and the stabilizer of $\mathfrak{S}_\lambda d\mathfrak{S}_\mu$ in $W(\lambda)\times W(\mu)$ is $\tilde{\Delta}_d(W(\lambda,\mu,d))$. Moreover, if d' is an element of $X_{\lambda\mu}$ such that $\mathfrak{S}_\lambda d\mathfrak{S}_\mu$ and $\mathfrak{S}_\lambda d'\mathfrak{S}_\mu$ are in the same $(W(\lambda)\times W(\mu))$-orbit, then $\mathfrak{S}_{\lambda\cap{}^d\mu}$ and $\mathfrak{S}_{\lambda\cap{}^{d'}\mu}$ are conjugate in $\mathcal{N}_\lambda$. Therefore,

$$\varphi_\lambda\varphi_\mu = \sum_{d \in X'_{\lambda\mu}} \frac{|W(\lambda)|.|W(\mu)|}{|W(\lambda,\mu,d)|}\varphi_{\lambda\cap{}^d\mu}, \tag{6}$$

where $X'_{\lambda\mu}$ denotes a cross-section of $\mathcal{N}_\lambda\backslash\mathfrak{S}_n/\mathcal{N}_\mu$ contained in $X_{\lambda\mu}$.

The descending Loewy series of $\bar{\mathcal{R}}_n$. We can now state the main results of this paper.

THEOREM A. *If $i \geqslant 0$, we have* $\left(\mathrm{Rad}\,\bar{\mathcal{R}}_n\right)^i = \underset{\lambda\in\mathrm{Part}_i^{(p)}(n)}{\oplus} \mathbb{F}_p\bar{\varphi}_\lambda.$

COROLLARY B. *The Loewy length of $\bar{\mathcal{R}}_n$ is* $[n/p]+1$.

Corollary B follows immediately from Theorem A. The end of this paper is devoted to the proof of Theorem A.

Proof of Theorem A. Let $\bar{\mathcal{R}}_n^{(i)} = \bigoplus_{\lambda \in \mathrm{Part}_i^{(p)}(n)} \mathbb{F}_p \bar{\varphi}_\lambda$. Note that

$$0 = \bar{\mathcal{R}}_n^{([n/p]+1)} \subset \bar{\mathcal{R}}_n^{([n/p])} \subset \cdots \subset \bar{\mathcal{R}}_n^{(1)} \subset \bar{\mathcal{R}}_n^{(0)} = \bar{\mathcal{R}}_n.$$

Let us first prove the following fact:

$$(\clubsuit) \qquad \textit{If } i,\ j \geqslant 0, \textit{ then } \bar{\mathcal{R}}_n^{(i)} \bar{\mathcal{R}}_n^{(j)} \subset \bar{\mathcal{R}}_n^{(i+j)}.$$

Proof of ($\clubsuit$). Let λ and μ be two compositions of n such that $\pi_p(\lambda) \geqslant i$ and $\pi_p(\mu) \geqslant j$. Let $d \in X'_{\lambda\mu}$ be such that p does not divide $\dfrac{|W(\lambda)|.|W(\mu)|}{|W(\lambda,\mu,d)|}$. By (6), we only need to prove that this implies that $\pi_p(\lambda \cap {}^d\mu) \geqslant i+j$. But our assumption on d means that $\tilde{\Delta}_d(W(\lambda,\mu,d))$ contains a Sylow p-subgroup of $W(\lambda) \times W(\mu)$. In particular, the p-rank of $W(\lambda,\mu,d)$ is greater than or equal to the p-rank of $W(\lambda) \times W(\mu)$. By (3), this means that the p-rank of $W(\lambda,\mu,d)$ is $\geqslant i+j$. Since $W(\lambda,\mu,d)$ is a subgroup of $W(\lambda \cap {}^d\mu)$, we get that the p-rank of $W(\lambda \cap {}^d\mu)$ is $\geqslant i+j$. In other words, again by (3), we have $\pi_p(\lambda \cap {}^d\mu) \geqslant i+j$, as desired. □

By ($\clubsuit$), $\bar{\mathcal{R}}_n^{(i)}$ is an ideal of $\bar{\mathcal{R}}_n$ and, if $i \geqslant 1$, then $\bar{\mathcal{R}}_n^{(i)}$ is a nilpotent ideal of $\bar{\mathcal{R}}_n$. Therefore, $\bar{\mathcal{R}}_n^{(1)} \subset \mathrm{Rad}\, \bar{\mathcal{R}}_n$. In fact:

$$(\diamondsuit) \qquad \mathrm{Rad}\, \bar{\mathcal{R}}_n = \bar{\mathcal{R}}_n^{(1)}.$$

Proof of ($\diamondsuit$). First, note that $\mathrm{Rad}\, \bar{\mathcal{R}}_n$ consists of the nilpotent elements of $\bar{\mathcal{R}}_n$ because $\bar{\mathcal{R}}_n$ is commutative. Now, let φ be a nilpotent element of $\bar{\mathcal{R}}_n$. Write $\varphi = \sum_{\lambda \in \mathrm{Part}(n)} a_\lambda \bar{\varphi}_\lambda$ and let $\lambda_0 \in \mathrm{Part}(n)$ be maximal (for the order $\leqslant$ on $\mathrm{Part}(n)$) such that $a_{\lambda_0} \neq 0$. Then, by (6), the coefficient of φ_{λ_0} in φ^r is equal to $a_{\lambda_0}^r |W(\lambda_0)|^{r-1}$. Therefore, since φ is nilpotent and $a_{\lambda_0} \neq 0$, we get that p divides $|W(\lambda_0)|$, so that $\lambda_0 \in \mathrm{Part}_1^{(p)}(n)$ (by (3)). Consequently, $\varphi - a_{\lambda_0}\bar{\varphi}_{\lambda_0}$ is nilpotent and we can repeat the argument to find finally that $\varphi \in \bar{\mathcal{R}}_n^{(1)}$. □

We shall now establish a special case of (5) (or (6)). We need some notation. If $\alpha = (\alpha_1, \ldots, \alpha_r)$ is a composition of n' and $\beta = (\beta_1, \ldots, \beta_s)$ is a composition of n'', let $\alpha \sqcup \beta$ denote the composition of $n'+n''$ equal to $(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r)$. If $1 \leqslant j \leqslant n$ and if $0 \leqslant k \leqslant [n/j]$, we denote by $\boldsymbol{\nu}(n,j,k)$ the composition $(n-jk, j, j, \ldots, j)$ of n, where j is repeated k times (if $n = jk$, then the part $n-jk$ is omitted). If $\lambda \in \mathrm{Comp}(n)$, we set

$$M(\lambda) = \{0\} \cup \{1 \leqslant j \leqslant n \mid p \text{ does not divide } r_j(\lambda)\},$$
$$\mathbf{m}(\lambda) = \max M(\lambda),$$
$$J(\lambda) = \{0\} \cup \{1 \leqslant j \leqslant n \mid r_j(\lambda) \geqslant p\},$$
$$\mathbf{j}(\lambda) = \min J(\lambda)$$

and

$$\mathbf{jm}(\lambda) = (\mathbf{j}(\lambda), \mathbf{m}(\lambda)).$$

Let $I = \{0, 1, \ldots, [n/p]\}$. Then $\mathbf{jm}(\lambda) \in I \times I$. Let us now introduce an order $\preccurlyeq$ on $I \times I$. If (j,m), (j',m') are two elements of $I \times I$, we write $(j,m) \preccurlyeq (j',m')$ if one of the following two conditions is satisfied:

(a) $j < j'$.
(b) $j = j'$ and $m \geqslant m'$.

Now, let $i \geqslant 1$ and let $\lambda \in \mathrm{Part}_{i+1}^{(p)}(n)$. Let $(j,m) = \mathbf{jm}(\lambda)$. Then $\lambda = \widehat{\alpha \sqcup \nu_0}$, where α is a partition of $n-m-jp$ and $\nu_0 = \boldsymbol{\nu}(m+jp, j, p)$. Let $\tilde{\lambda} = \alpha \sqcup (m+jp)$.

Then $\pi_p(\tilde{\lambda}) = i$ (indeed, $r_{m+jp}(\tilde{\lambda}) = 1 + r_{m+jp}(\alpha) = 1 + r_{m+jp}(\lambda)$ and, by the maximality of m, we have that p divides $r_{m+jp}(\lambda)$) and

$$(\heartsuit) \qquad \bar{\varphi}_{\boldsymbol{\nu}(n,j,p)}\bar{\varphi}_{\tilde{\lambda}} \in \bar{\varphi}_\lambda + \Bigl(\bigoplus_{\substack{\mu\in\mathrm{Part}_{i+1}^{(p)}(n)\\ \mathbf{jm}(\mu)\prec(j,m)}} \mathbb{F}_p\bar{\varphi}_\mu\Bigr).$$

Proof of $(\heartsuit)$. Let $\nu = \boldsymbol{\nu}(n,j,p)$. Since $\pi_p(\lambda) = i+1 \geqslant 2$, there exists $j' \in \{1,2,\dots,[n/p]\}$ such that $r_{j'}(\lambda) \geqslant p$. Then $j' \geqslant j$ (by definition of j), so $n \geqslant 2pj$. In particular, $n - pj > j$, so $W(\nu) \simeq \mathfrak{S}_p$. Now, if $m' > m$, then $r_{m'}(\alpha) = r_{m'}(\lambda)$, so

$$(\heartsuit') \qquad \forall\ m' > m,\ r_{m'}(\alpha) \equiv 0 \mod p.$$

Also

$$(\heartsuit'') \qquad \forall\ l \neq m + jp,\ r_l(\tilde{\lambda}) = r_l(\alpha)$$

and

$$(\heartsuit''') \qquad r_{m+jp}(\tilde{\lambda}) = r_{m+jp}(\alpha) + 1.$$

Now, keep the notation of (6). We may, and we will, assume that $1 \in X'_{\nu\tilde{\lambda}}$. First, note that $\nu \cap \tilde{\lambda} = \alpha \sqcup \nu_0$ and that the image of $\tilde{\Delta}_1$ in $W(\nu) \times W(\tilde{\lambda})$ is equal to $W(\nu) \times W(\alpha)$. But, by $(\heartsuit')$, $(\heartsuit'')$ and $(\heartsuit''')$, the index of $W(\alpha)$ in $W(\tilde{\lambda})$ is $\equiv 1$ mod p. Thus, by (6), we have

$$\bar{\varphi}_\nu\bar{\varphi}_{\tilde{\lambda}} = \bar{\varphi}_\lambda + \sum_{d\in X'_{\nu\tilde{\lambda}}-\{1\}} \frac{|W(\nu)|.|W(\tilde{\lambda})|}{|W(\nu,\tilde{\lambda},d)|}\bar{\varphi}_{\nu\cap{}^d\tilde{\lambda}}.$$

Now, let d be an element of $X_{\nu\tilde{\lambda}}$ such that p does not divide $\dfrac{|W(\nu)|.|W(\tilde{\lambda})|}{|W(\nu,\tilde{\lambda},d)|} = x_d$ and such that $\mathbf{jm}(\nu \cap {}^d\tilde{\lambda}) \not\prec \mathbf{jm}(\lambda)$. It is sufficient to show that $d \in \mathcal{N}_\nu\mathcal{N}_{\tilde{\lambda}}$. Write $\alpha = (\alpha_1,\dots,\alpha_r)$. Then

$$\nu \cap {}^d\tilde{\lambda} = (n_1,\dots,n_r,n_0) \sqcup j^{(1)} \sqcup \cdots \sqcup j^{(p)},$$

where $n_k \geqslant 0$ and $j^{(l)}$ is a composition of j with at most $r+1$ parts. Since p does not divide x_d, the image of $\mathcal{N}_\nu \cap {}^d\mathcal{N}_{\tilde{\lambda}}$ in $W(\nu)$ contains a Sylow p-subgroup of $W(\nu) \simeq \mathfrak{S}_p$. Let $w \in \mathcal{N}_\nu \cap {}^d\mathcal{N}_{\tilde{\lambda}}$ be such that its image in $W(\nu)$ is an element of order p. Then there exists $\sigma \in \mathfrak{S}_\nu$ such that $w\sigma$ normalizes $\mathfrak{S}_{\nu\cap{}^d\tilde{\lambda}}$. In particular, $\widehat{j^{(1)}} = \cdots = \widehat{j^{(p)}}$. So, if $j^{(1)} \neq (j)$, then $\mathbf{j}(\nu \cap {}^d\tilde{\lambda}) < \mathbf{j}(\lambda)$, which contradicts our hypothesis. So $j^{(1)} = \cdots = j^{(p)} = (j)$. Therefore,

$$\tilde{\lambda} \cap {}^{d^{-1}}\nu = \boldsymbol{\nu}(\alpha_1,j,k_1) \sqcup \cdots \sqcup \boldsymbol{\nu}(\alpha_r,j,k_r) \sqcup \boldsymbol{\nu}(m+jp,j,k_0),$$

where $0 \leqslant k_i \leqslant p$ and $\sum_{i=0}^r k_i = p$. Note that $(n_1,\dots,n_r,n_0) = (\alpha_1 - k_1j,\dots,\alpha_r - k_rj,\alpha_0 - k_0j)$ where, for simplification, we denote $\alpha_0 = m+jp$. Also, $\mathbf{j}(\tilde{\lambda}\cap{}^{d^{-1}}\nu) \leqslant j$ and, since $\mathbf{jm}(\tilde{\lambda} \cap {}^{d^{-1}}\nu) \not\prec (j,m)$, we have that $\mathbf{m}(\tilde{\lambda} \cap {}^{d^{-1}}\nu) \leqslant m$. Recall that $d^{-1}wd \in \mathcal{N}_{\tilde{\lambda}}$. So two cases may occur:

- Assume that there exists a sequence $0 \leqslant i_1 < \cdots < i_p \leqslant r$ such that $0 \neq k_{i_1} = \cdots = k_{i_p}$ $(= 1)$ and such that $\alpha_{i_1} = \cdots = \alpha_{i_p}$. So $r_l(\tilde{\lambda} \cap {}^{d^{-1}}\nu) \equiv r_l(\tilde{\lambda}) \mod p$ for every $l \geqslant 1$. In particular, $r_{m+jp}(\tilde{\lambda}\cap{}^{d^{-1}}\nu) \equiv 1 + r_{m+jp}(\alpha) \equiv 1 \mod p$ by $(\heartsuit')$ and $(\heartsuit''')$. Thus, $\mathbf{m}(\tilde{\lambda} \cap {}^{d^{-1}}\nu) \geqslant m + jp > m$, which contradicts our hypothesis.
- So there exists a unique $i \in \{0,1,\dots,r\}$ such that $k_i = p$. Consequently, $k_{i'} = 0$ if $i' \neq i$. If $\alpha_i > m+jp$, then $r_{\alpha_i}(\tilde{\lambda}\cap{}^{d^{-1}}\nu) = r_{\alpha_i}(\tilde{\lambda})-1 = r_{\alpha_i}(\alpha)-1$ (by $(\heartsuit'')$), so p does not divide $r_{\alpha_i}(\tilde{\lambda}\cap{}^{d^{-1}}\nu)$ (by $(\heartsuit')$), which implies that $\mathbf{m}(\tilde{\lambda}\cap{}^{d^{-1}}\nu) \geqslant \alpha_i > m$,

contrarily to our hypothesis. If $\alpha_i < m+jp$, then $r_{m+jp}(\tilde{\lambda} \cap {}^{d^{-1}}\nu) = r_{m+jp}(\tilde{\lambda}) = r_{m+jp}(\alpha)+1$ (by ($\heartsuit'''$)), so p does not divide $r_{m+jp}(\tilde{\lambda} \cap {}^{d^{-1}}\nu)$ (by ($\heartsuit'$)), contrarily to our hypothesis. This shows that $\alpha_i = m+jp$. In other words, $d \in \mathcal{N}_{\tilde{\lambda}}\mathcal{N}_\nu$, as desired. $\square$

By ($\diamondsuit$), Theorem A follows immediately from the next result: if $i \geqslant 0$, then

$$(\spadesuit) \qquad \bar{\mathcal{R}}_n^{(1)}\bar{\mathcal{R}}_n^{(i)} = \bar{\mathcal{R}}_n^{(i+1)}.$$

Proof of ($\spadesuit$). We may assume that $i \geqslant 1$. By ($\clubsuit$), we have $\bar{\mathcal{R}}_n^{(1)}\bar{\mathcal{R}}_n^{(i)} \subset \bar{\mathcal{R}}_n^{(i+1)}$. So we only need to prove that, if $\lambda \in \mathrm{Part}_{i+1}^{(p)}(n)$ then $\bar{\varphi}_\lambda \in \bar{\mathcal{R}}_n^{(1)}\bar{\mathcal{R}}_n^{(i)}$. But this follows from ($\heartsuit$) and an easy induction on $\mathbf{jm}(\lambda) \in I \times I$ (for the order $\preccurlyeq$). $\square$

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